- **Clairaut's Theorem** or **Schwarz's Theorem**: Suppose $f(x, y)$ is defined throughout an open disk *D* centered at (a,b). If f_{xy} and f_{yx} are both continuous on *D*, then $f_{xy}(a,b) = f_{yx}(a,b)$.
  - Extend to *n* variables and multi-order partial differentiation.
- **Hessian Matrix**
  - $\mathbf{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \end{bmatrix}$
  - Matrix of all second-order partial derivatives of a scalar multivariable function $f(x_1, x_2, ..., x_n)$
  - In most cases, **H** is diagonally symmetrical due to Clairaut's Theorem
  - **H** is just the Jacobian of a Jacobian $\mathbf{J} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$ ($m = 1$ case)
  - **Hessian**: det(**H**)
  - Order of variables does not matter as long as you are consistent!
- **Second Derivatives Test**
  - Conditions:
    - All second partial derivatives of $f(x_1, x_2, ..., x_n)$ are continuous on an *n*-disk centered at $(a_1, a_2, ... a_n)$ such that $\nabla f(a_1, a_2, ... a_n) = \vec{0}$.
    - Suppose *k* is $\{k \in \mathbb{Z}^+ \mid k \le n\}$
    - Let $D_k = \det(\mathbf{H})$ in the variables $x_1, x_2, ..., x_k$
  - If $D_k(a_1, a_2, ... a_n) > 0$ for all *k*, then $f(x_1, x_2, ..., x_n)$ has a local minimum at $(a_1, a_2, ... a_n)$.
  - Else if $(-1)^k D_k(a_1, a_2, ... a_n) > 0$ for all *k*, then $f(x_1, x_2, ..., x_n)$ has a local maximum at $(a_1, a_2, ... a_n)$.
  - Else if $D_k(a_1, a_2, ... a_n) \ne 0$, then $f(x_1, x_2, ..., x_n)$ has a saddle point at $(a_1, a_2, ... a_n)$.
  - Else that $D_k(a_1, a_2, ... a_n) = 0$, then the second derivatives test is inconclusive.
- **Second-Degree Taylor Polynomial**
  - The second-degree Taylor polynomial of $f(\vec{x}) = f(x_1, x_2, ..., x_n)$ centered about $\vec{a} = (a_1, a_2, ... a_n)$ is $P_2(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) + \frac{1}{2}(\vec{x} - \vec{a})^T Hf(\vec{a})(\vec{x} - \vec{a})$ where $Hf(\vec{a})$ is the Hessian at $f(\vec{a})$. $(\vec{x} - \vec{a})$ denotes the matrix column picture of the vector $\vec{x} - \vec{a}$. Note that these involve matrix operations!